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Computational Methods for Solving General Harmonic-like Quasi Variational Inequalities

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ABSTRACT
Some classes of harmonic-like quasi variational inequalities, which can be viewed as novel important generalization of quasi variational equalities, are introduced and investigated. Using various techniques such as projection methods, auxiliary principle, dynamical systems coupled with finite difference approach, we suggest and analyze a number of new and known numerical techniques for solving quasi harmonic-like variational inequalities. Convergence analysis of these methods is investigated under suitable conditions. Sensitivity analysis is also investigated. One can obtain a number of new classes of quasi harmonic-like variational inequalities by interchanging the role of operators. Various special cases are discussed as applications of the main results. Several open problems are suggested for future research.

KEYWORDS
Variational inequalities, Convex functions, fixed point, iterative methods, auxiliary principle, convergence, dynamical systems, sensitivity analysis

1. Introduction

Variational inequality theory contains a wealth of new ideas and techniques. Variational inequality theory was introduced and considered in early sixties by Lions and Stampacchia[71], can be viewed as a novel extension and generalization of the variational principles. By the variational principles, we mean maximum and minimum problems arising in game theory, mechanics,geometrical optics, general relativity theory, economics, transportation, differential geometry and related areas It is amazing that a wide class of unrelated problems can be studied in the general and unified framework of variational inequalities. Variational inequalities have been generalized and extended in several directions using novel and innovative ideas to handle complicated and complex problems. Noor [41, 42] considered two new classes of variational inequalities involving two arbitrary operators in 1988, which are known as general variational inequalities and have applications in oceanography, non-positive and non symmetric differential equations theory. An important special case of these general variational inequalities is known as inverse variational inequalities, see [19, 23, 60, 73, 75] for more details.

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It is worth mentioning that variational inequalities theory is clasely related to the convexity theory contains a wealth of novel ideas and techniques. Several new generalizations and extensions of the convex functions and convex sets have been introduced and studied to tackle unrelated complicated and complex problems in a unified manner. Harmonic functions and harmonic convex sets are important generalizations of the convex functions andconvex sets. Anderson et al. [3] have investigated several aspects of the harmonic convex sets and harmonic convex functions, which can viewed as important generalizations of the convex functions and convex sets. The harmonic means have novel applications in electrical circuits theory, signal processing and semiconductor physics. It is known that the total resistance of a set of parallel resistors is obtained by adding up the reciprocals of the individual resistance values, and then taking the reciprocal of their total. More precisely, if η_1 and η_2 are the resistances of two parallel resistors, then the total resistance is computed by the formula $(\frac{1}{\eta_1} + \frac{1}{\eta_2})^{-1} = \frac{\eta_1\eta_2}{\eta_1+\eta_2}$, which is half the harmonic means. The harmonic mean is employed in finance to determine the average multiples like the price-income ratio. Al-Azemi et al. [8] studied the Asian options with harmonic average, which can be viewed a new direction in the study of the risk analysis stock exchange and financial mathematics. The harmonic mean are being used to suggest some iterative methods for solving nonlinear equations. Noor et al. [61, 62] have shown that the minimum of the differentiable harmonic convex functions on the harmonic convex set can be characterized by a class of harmonic variational inequalities. Noor et al. [57–59] introduced the new concepts of harmonic-like convex sets and harmonic-like convex functions. Harmonic convexity also has implications in signal processing. Functions that exhibit harmonic convexity are associated with higher frequency components that can distort fundamental waveforms, leading to undesired oscillations. This property is particularly relevant in applications in waveform stability, signal transmission, machine learning, artificial intelleigence, data science, entropy analysis and informaion technolongoy.

It has been established that the variational inequalities are equivalent to the fixed point problems. This equivalent formulations have played an important role to study the existence of the solution and to develop efficient numerical methods for solving variational inequalities and related optimization problems. Noor [45, 46] has proposed and suggested three step forward-backward iterative methods, known as Noor iterations, for finding the approximate solution of general variational inequalities using the technique of updating the solution and auxiliary principle. These forward-backward splitting algorithms are similar to those of the schemes of Glowinski and Le Tallec [22], which they suggested by using the Lagrangian technique and Tseng [74]. Suantai et. al. [72] have also considered some novel forward-backward algorithms for optimization and their applications to compressive sensing and image inpainting. Ashish [4], Ashish et. al. [5–7], Cho et al. [13], Chugh et al. [14]Kwuni et al. [28] and Natarajan et al. [33] and Yadav et al. [79] explored the Julia set and Mandelbrot set in Noor orbit using the Noor (three step) iterations. It is worth mentioning that Noor iterations have influenced the research in the fixed point theory, data analysis, infomation science, financial mathematics, green energy, solar system, medical sciences and will continue to inspire further research in fractal geometry, chaos theory, coding, number theory, spectral geometry, dynamical systems, complex analysis, nonlinear programming, graphics and computer aided design. These three-step schemes are a natural generalization of the splitting methods for solving partial differential equations. Noor (three-step) iterations contain Mann (one-step) iteration and Ishikawa (two-step) iterations as special cases. Inspired and motivated by the usefulness and applications of the splitting three-step methods, several classes of three-step approximation schemes for solving variational inequalities, fixed points and related problems are being investigated. It has been established [11, 12] that Noor (Three step) iterations, perform better than the (Ishikawa (two step) iterations and one step method Mann (one step) iteration.

In various cases, the underlying set may depends upon the solution explicitly or implicity, then the variational inequality is called the quasi-variational inequality. Bensoussan and Lions [10] studied such type of problems in the field of impulse control. To be more precise, for given nonlinear operators $\mathcal{T}, g : \mathcal{H} \leftarrow \mathcal{H}$ and convex valued closed convex set $\Omega(\mu)$, find $\mu \in \Omega(\mu) \subseteq \mathcal{H}$, such that

$$\langle \mathcal{T}\mu, \nu - g(\mu)\rangle \geq 0, \quad \forall \nu \in \Omega(\mu), \tag{1}$$

which is called the general quasi variational inequality considered by Noor [42] in 1988. If $\mathcal{T} = I$, then the problem (1) has been used to consider a road pricing problem in which the environment impact problem due to traffic flow is taken into account to fix the road taxes and a bipartite market equilibrium problems. Also, a number of important and complictated problems such as flow control problems, transportation, telecommunication networks, dynamic power price problem [73] and least distance problem [11] can be studied in the unified framework of general quasi variational inequalities.

Noor [40, 44, 46] proved that the quasi variational inequalities are equivalent to the implicit fixed point problem. This equivalent formulation played an important role in developing numerical methods, sensitivity analysis, dynamical systems and other aspects of quasi-variational inequalities. For the applications, motivations, generalizations, extensions of quasi variational inequalities and related problems such as dynamical systems, sensitivity analysis, numerical methods, error bounds and related optimization programming problems, see [1, 2, 8, 10, 11, 19–24, 26–30, 32–35, 35–45, 45–53, 55, 61, 62, 66–77] and the references therein.

The projected dynamical systems associated with variational inequalities were considered by Dupuis and Nagurney [32]. The novel feature of the projected dynamical system is that the its set of stationary points corresponds to the set of the corresponding set of the solutions of the variational inequality problem. This dynamical system is a first order initial value problem. Consequently, equilibrium and nonlinear problems arising in various branches in pure and applied sciences can now be studied in the setting of dynamical systems. It has been shown [2, 19, 20, 26, 32, 47, 53–55, 59, 63–65, 74–77] that the dynamical systems are useful in developing some efficient numerical techniques for solving variational inequalities and related optimization problems.

Motivated and inspired by ongoing recent research in harmonic-like variational inequalities, we consider and study the general harmonic-like quasi variational inequalities, involving two arbitrary operators. We establish the equivalence between the harmonic-likeequasi variational inequalities and fixed point problem, which has been used to consider an iterative method for solving harmonic-like quasi variational inequalities. We prove that the nonlinear programming problems and implicit second order obstacle boundary value problems can be studied via the harmonic-like quasi variational inequalities. Several special cases are discussed as applications of the harmonic-like quasi variational inequalities in Section 2. In section 3, we discuss the unique existence of the solution as well as to suggest several inertial iterative method along

with the convergence analysis. We also apply the auxiliary principle technique involving an arbitrary operator is used to discuss some iterative schemes for solving the harmonic-like quasi variational inequalities in Section 4. In section 5, the dynamical system approach is applied to study the stability of the solution and to suggest some iterative methods exploring the finite difference idea. Our results in this section can be viewed as significant refinement of the known results.
One of the main purposes of this paper is to demonstrate the close connection among various classes of algorithms for the solution of the harmonic-like quasi variational inequalities and to point out that researchers in different field of variational inequalities and optimization. We would like to emphasize that the results obtained and discussed in this paper may motivate and bring a large number of novel, innovate potential applications, extensions and interesting topics in these areas. We have given only a brief introduction of this fast growing field. The interested reader is advised to explore this field further and discover novel and fascinating applications of general harmonic-like quasi variational inequalities in other areas of sciences such as machine learning, artificial intelligence, data analysis, fuzzy systems, random stochastic, financial analysis and related other optimization problems. It is expected the techniques and ideas of this paper may be starting point for further research.

2. Formulations and basic facts

Let Ω be a nonempty closed set in a real Hilbert space $\mathcal{H}$. We denote by $\langle\cdot,\cdot\rangle$ and $\|\cdot\|$ be the inner product and norm, respectively. First of all, we recall some concepts from convex analysis [3, 16–18, 35, 59] , which are needed in the derivation of the main results.

Definition 2.1. *[57, 59] A set Ω in $\mathcal{H}$ is said to be a harmonic-like convex set, if*

$$(\frac{2w\mu}{w+\mu})+\lambda(\nu-\mu)\in\Omega, \qquad \forall w,\mu,\nu\in\Omega,\lambda\in[0,1], \quad \mu\leq w\leq\nu.$$

Definition 2.2. *A function Φ is said to be a convex function, if*

$$\Phi(\frac{2w\mu}{w+\mu})+\lambda(\nu-\mu))\leq(1-\lambda)\Phi(\mu)+\lambda\Phi(\nu), \quad \forall w,\mu,\nu\in\Omega, \quad \lambda\in[0,1].$$

Remark 2.1. *The elemnt $w\in\Omega$ can be regarded as the weight function. For example, if $w=\nu$, then the harminc-lke convex set and harmonc-like convex functions becomes harmonic mean set and harmonic mean convex functions, which appears to be new concepts. For $w=\mu$, the harmic-like set and harmonic-like convex functions reduces to classical convex set and convex function.*

Convex functions are closely related to the integral inequalities and variational inequalities. These type of inequalities have played crucial part in developing fields such as: numerical analysis, operations research, transportation, financial mathematics, structural analysis, dynamical systems, sensitivity analysis and machine learning. Noor and Noor [57, 59] have shown that, if Φ is differentiable harmonic-like convex function, then $\mu\in\Omega$ is the minimum of the function Φ, if and only if, $\mu\in\Omega$ satisfies the inequality

$$\langle \Phi'(\frac{2w\mu}{w+\mu}), \nu - \mu \rangle \geq 0, \quad \forall w, \nu \in \Omega. \tag{2}$$

The inequalities of the type (2) are called the harmonic-like variational inequalities. It is known that the problems (2) occur, which may not be derivative of the differentiable functions. These facts and observations motivated us to consider more general variational inequalities of which (2) is a special case.
To be more precise, for a given nonlinear operator $\mathcal{T} : \mathcal{H} \longrightarrow \mathcal{H}$, we consider the problem of finding $\mu \in \Omega$ such that

$$\langle \mathcal{T}(\frac{2w\mu}{w+\mu}), \nu - \mu \rangle \geq 0, \quad \forall w, \nu \in \Omega. \tag{3}$$

which is called the harmonic-like variational inequality. Note that, for $\Phi'(\frac{2w\mu}{w+\mu}) = \mathcal{T}(\frac{2w\mu}{w+\mu})$, problem (3) is exactly the problem (2).
If $w = \mu$, then the problem (3) reduces to finding $\mu \in \Omega$ such that

$$\langle \mathcal{T}\mu, \nu - \mu \rangle \geq 0, \quad \forall w, \nu \in \Omega. \tag{4}$$

which is known as the vriational inequality, introduced and studied by Lions and Stamapacchia [29].

In many cases, the set and function may not be a convex set and convex functions. To overcome these drawbacks, Noor [49, 50] introduced the concept of general new general convex set and general convex function with respect to an arbitrary function g. For the sake of completeness and to convey an idea of this result, we include some details.

Definition 2.3. *[49, 50] A set $\Omega_g \subseteq \mathcal{H}$ is said to be a general harmonic-like convex set, if there exists an arbitrary function $g : \mathcal{H} \longrightarrow \mathcal{H}$ such that*

$$g(\frac{2w\mu}{w+\mu}) + t(g(\nu) - g(\mu)) \in \Omega_g, \quad \forall \mu, w, \nu \in \Omega_g. \quad t \in [0,1], \quad w \in [\mu, \nu].$$

Note that every harmonic-like convex set is a general harmonic convex set, but the converse is not true. It is worth mentioning that, for $w = \mu$, we obtain a new general convex set, which is different than the E-convex set of Youness [78] and various other general convex sets of Noor [**?**]. For the applications of the general convex sets in information technology, railway systems, computer aided design, digital vector optimization and comparison with other concepts, see [17, 18]. If $g = I, w = \mu$, then the general convex set Ω_g is exactly the convex set Ω.

Definition 2.4. *The function $\Phi : \Omega_g \longrightarrow \mathcal{H}$ is said to be general harmonic-like convex, if there exists an arbitrary function g, such that*

$$\Phi(g(\frac{2w\mu}{w+\mu}) + t(g(\nu) - g(\mu))) \leq \Phi(g(\mu)) + t\{\Phi(g(\nu) - \Phi(g(\mu))\}, \quad \forall \mu, w, \nu \in \Omega_g, \quad t \in [0,1].$$

For $w = \mu$, the general harmonic-like convex function reduces to:

$$\Phi((g(\mu) + t(g(\nu) - g(\mu))) \leq \Phi(g(\mu)) + t\{\Phi(g(\nu) - \Phi(g(\mu))\}, \quad \forall \mu, \nu \in \Omega_g, \quad t \in [0, 1],$$

which is also called the general convex function and appears to be a new one.

For the differentiable general harmonic-like convex function, we have

Theorem 2.1. *[63, 64] Let Φ be a differentiable general harmonic-like convex function on the general harmonic-like convex set Ω_g. Then the minimum $\mu \in \Omega_g$ of the function Φ, if and only if, $\mu \in \Omega_g$ satisfies the inequality*

$$\langle \Phi'(g(\frac{2w\mu}{w+\mu})), g(\nu) - g(\mu)\rangle \geq 0, \quad \forall w, \nu \in \Omega_g, \tag{5}$$

where $\Phi'(.)$ is the differential of Φ at $\mu \in \Omega_g$.

Theorem 2.1 implies that general convex programming problems can be studied via the general variational inequality(5) .
It is known that the inequality of the type (5) may not arise as the optimality condition of the differentiable harmonic-like convex functions.
Noor [40] introduced and investigated the problem of finding $\mu \in \Omega \subseteq \mathcal{H}$, such that

$$\langle \mathcal{T}(\frac{2w\mu}{w+\mu}), g(\nu) - g(\mu)\rangle \geq 0, \quad \forall \nu, w \in \Omega, \tag{6}$$

which is called the general harmonic-like variational inequalities. For the applications, motivations, numerical results, dynamical systems and related optimizations, see [63, 64].

In many applications, the convex set Ω may depend upon the solution explicitly or implicitly. In such cases, variational inequality is called the quasi variational inequality. Let $\Omega : \mathcal{H} \longrightarrow \mathcal{H}$ be a set-valued mapping which, for any element $\mu \in \mathcal{H}$, associates a convex-valued and closed set $\Omega(\mu) \subseteq \mathcal{H}$. We now consider some new classes of general harmonic-like quasi variational inequalities, which include several new and known classes of variational inequalities as special cases.

For given nonlinear operators $\mathcal{T}, g, h$, we consider the problem of finding $\mu \in \Omega(\mu)$, such that

$$\langle \mathcal{T}(\frac{2w\mu}{w+\mu}), h(\nu) - g(\mu)\rangle \geq 0, \quad \forall w, \nu \in \Omega(\mu), \quad \mu \leq w \leq \nu, \tag{7}$$

which is called the general harmonic-like quasi variational inequality.
if $w = \nu$, then the problem (7) collapses to finding $\mu \in \Omega(\mu)$ such that

$$\langle \mathcal{T}(\frac{2w\nu}{w+\nu}), h(\nu) - g(\mu)\rangle \geq 0, \quad \forall w, \nu \in \Omega(\mu), \quad \mu \leq w \leq \nu, \tag{8}$$

which is called the new general harmonic mean quasi variational inequality.

For $w = \mu$, the problem (7) reduces to finding $\mu \in \Omega(\mu)$, such that

$$\langle \mathcal{T}\mu, h(\nu) - g(\mu)\rangle \geq 0, \quad \forall w, \nu \in \Omega(\mu), \tag{9}$$

which is known as general quasi variational inequality and appears to be a new.
If $g = I$, the identity operator, then problem (9) collapses to to findg $\mu \in \Omega(\mu)$ such that

$$\langle \mathcal{T}\mu, h(\nu) - \mu\rangle \geq 0, \quad \forall w, \nu \in \Omega(\mu), \tag{10}$$

is the called quasi variational inequality, which is different than the quasi variational inequality introduced and studied by Bensoussan and Lions [10].
Also by interchanging the role of the operators $\mathcal{T}$ and g, the problem (7) is equivalent to finding $\mu \in \Omega(\mu)$, such that

$$\langle g(\frac{2w\mu}{w+\mu}), h(\nu) - \mathcal{T}(\mu)\rangle \geq 0, \quad \forall w, \nu \in \Omega(\mu), \tag{11}$$

Note the symmetry role played by the mappings $\mathcal{T}$ and g. It is clear all the results, which hold for the problem (7), continue to hold for the problem (11) and vice versa.
If $\mathcal{T} = I$, the identity operator, then the problem (11) reduces to finding $\mu \in \Omega(\mu)$, such that

$$\langle (\frac{2w\mu}{w+\mu}), h(\nu) - g(\mu)\rangle \geq 0, \quad \forall \nu \in \Omega(\mu), \tag{12}$$

is called the inverse general harmonic-like quasi variational inequality.
For $g = I$, the problem (11) reduces to finding $\nu \in \Omega(\mu)$, such that

$$\langle (\frac{2w\mu)}{w+\mu}), h(\nu) - \mathcal{T}(\mu)\rangle \geq 0, \quad \forall \nu \in \Omega(\mu), \tag{13}$$

which is also called the inverse quasi harmonic-like variational inequality. Consequently, it is evident that all the known results for quasi harmonic-like variational inequalities are also valid diferent types of inverse quasi harmonic-like variational inequalities. This is a surprising fascinating fact.

Special Cases

We now point out some very important and interesting problems, which can be obtained as special cases of the problem (7).

(I). This problem (7) can be viewed as a problem of finding the minimum of general convex function. Such type of problems have applications in optimization theory and imaging process in medical sciences and earthquake.

(II). If $\Omega^*(\mu) = \{\mu \in \mathcal{H} : \langle \mu, \nu\rangle \geq 0, \quad \forall \nu \in \Omega(\mu)\}$ is a polar (dual) cone of a convex-valued cone $\Omega(\mu)$ in $\mathcal{H}$, then problem (7) is equivalent to finding $\mu \in \mathcal{H}$, such that

$$g(\mu) \in \Omega(\mu), \quad \mathcal{T}(\frac{2w\mu)}{w+\mu}) \in \Omega^*(\mu) \quad \text{and} \quad \langle \mathcal{T}(\frac{2w\mu)}{w+\mu}), g(\mu)\rangle = 0, \tag{14}$$

which is known as the harmonic-like quasi complementarity problems and appears to be a new one.

(III). For $\Omega(\mu) = \Omega$, the convex set, the problem (14) is equivalent to finding $\mu \in \mathcal{H}$ such that

$$g(\mu) \in \Omega, \quad \mathcal{T}(\frac{2w\mu)}{w+\mu}) \in \Omega^* \quad \text{and} \quad \langle \mathcal{T}(\frac{2w\mu)}{w+\mu}), g(\mu)\rangle = 0, \tag{15}$$

is called the harmonic-like nonlinear complementarity problem. Clearly, for $\mu = w$, the problems (14) and (15) reduces to;

$$g(\mu) \in \Omega(\mu), \quad \mathcal{T}\mu \in \Omega^*(\mu) \quad \text{and} \quad \langle \mathcal{T}\mu, g(\mu)\rangle = 0, \tag{16}$$

quasi nonlinear complementarity problems and

$$g(\mu) \in \Omega, \quad \mathcal{T}\mu \in \Omega^* \quad \text{and} \quad \langle \mathcal{T}\mu, g(\mu)\rangle = 0, \tag{17}$$

nonlinear complemenatrity probelms respectively.

This shows that the harmonic-like quasi nonlinear complemenatrity problem (14) is quite general unifying ones and include the nonlineat complementarity problems and linear complementarity problems as special cases. The complementarity problems were introduced and studied by Cottle et al.[15], Liu et al. [30], Noor [41, 42] and Noor et al.[52, 64, 68] in game theory, management sciences and quadratic programming as special cases. This inter relationship among these problems have played a major role in developing numerical results for these problems and their applications.

(IV). If $\Omega(\mu) = \Omega$, where Ω is a convex set in $\mathcal{H}$, then problem (7) reduces to finding $\mu \in \Omega$ such that

$$\langle \rho\mathcal{T}\mu, g(\nu) - g(\mu)\rangle \geq 0, \quad \forall \nu \in \Omega, \tag{18}$$

is known as the general variational inequality, introduced and studied by Noor [41].

(V). If $\Omega(\mu) = \Omega, \quad g = I, h = I$, where Ω is a convex set in $\mathcal{H}$ and $\mathcal{T} = I$, then problem (18) reduces to finding $\mu \in \Omega$ such that

$$\langle \rho\mathcal{T}\mu, \nu - \mu\rangle \geq 0, \quad \forall \nu \in \Omega, \tag{19}$$

is known as the variational inequality, introduced and studied by Lions and Stampacchia [33].

(VI). If $\Omega(\mu) = \Omega$, where Ω is a convex set in $\mathcal{H}$ and $\mathcal{T} = I$, then problem (12) reduces to finding $\mu \in \Omega$ such that

$$\langle \rho\mu, h(\nu) - g(\mu)\rangle \geq 0, \quad \forall \nu \in \Omega, \tag{20}$$

is known as the inverse variational inequality. For the applications and numerical methods for solving the inverse variational inequalities, see [19, 23, 60, 73, 75] .

Remark 2.2. *It is worth mentioning that for appropriate and suitable choices of the operators $\mathcal{T}, g, h$, set-valued convex set $\Omega(\mu)$ and the spaces, one can obtain several classes of inverse variational inequalities, inverse complementarity problems and optimization problems as special cases of the harmonic-like quasi variational inequalities (7). This shows that the problem (7) is quite general and unifying one. It is interesting problem to develop efficient and implementable numerical methods for solving the harmonic-like quasi-variational inequalities and their variants.*

We also need the following result, known as the projection Lemma(best approximation), which plays a crucial part in establishing the equivalence between the harmonic-like quasi variational inequalities and the fixed point problems. This result is used in the analyzing the convergence analysis of the implicit and explicit methods for solving the variational inequalities and related optimization problems.

Lemma 2.1.1. *[2, 42] Let $\Omega(\mu)$ be a closed and convex-valued set in $\mathcal{H}$. Then, for a given $z \in \mathcal{H}$, $\mu \in \Omega(\mu)$ satisfies the inequality*

$$\langle \mu - z, \nu - \mu \rangle \geq 0, \quad \forall \nu \in \Omega(\mu), \tag{21}$$

if and only if,

$$\mu = \Pi_{\Omega(\mu)}(z),$$

where $\Pi_{\Omega(\mu)}$ is implicit projection of $\mathcal{H}$ onto the closed convex-valued set $\Omega(\mu)$.

It is well known that the implicit projection operator $\Pi_{\Omega(\mu)}$ is not nonexpansive, but it is required to satisfy the following assumption, which plays an important part in the derivation of the results..

Assumption 2.1.

$$\|\Pi_{\Omega(\mu)}\omega - \Pi_{\Omega(\nu)}\omega\| \leq \eta\|\mu - \nu\|, \forall \mu, \nu, \omega \in \mathcal{H}, \tag{22}$$

where $\eta > 0$ is a constant.

Assumption 2.1 has been used to prove the existence of a solution of general quasi variational inequalities as well as in analyzing convergence of the iterative methods.

Definition 2.5. *An operator $\mathcal{T} : \mathcal{H} \to \mathcal{H}$ is said to be:*

(1) Strongly harmonic-like monotone, if there exist a constant $\alpha > 0$, such that

$$\langle \mathcal{T}(\frac{2w\mu}{w+\mu}) - \mathcal{T}(\frac{2w\nu}{w+\nu}), \mu - \nu \rangle \geq \alpha\|\mu - \nu\|^2, \quad \forall w, \mu, \nu \in \mathcal{H}.$$

(2) harmonic-like Lipschitz continuous, if there exist a constant $\beta > 0$, such that

$$\|\mathcal{T}(\frac{2w\mu}{w+\mu}) - \mathcal{T}(\frac{2w\nu}{w+\nu})\| \leq \beta\|\mu - \nu\|, \quad \forall w, \mu, \nu \in \mathcal{H}.$$

(3) harmonic-like monotone, if

$$\langle \mathcal{T}(\frac{2w\mu}{w+\mu}) - \mathcal{T}(\frac{2w\nu}{w+\nu}), \mu - \nu \rangle \geq 0, \quad \forall w, \mu, \nu \in \mathcal{H}.$$

(4) harmonic-like pseudo monotone, if

$$\langle \mathcal{T}(\frac{2w\mu}{w+\mu}), \nu - \mu \rangle \geq 0 \quad \Rightarrow \quad \langle \mathcal{T}(\frac{2w\nu}{w+\nu}), \nu - \mu \rangle \geq 0, \quad \forall w, \mu, \nu \in \mathcal{H}.$$

For $w = \mu$, Definition 2 reduces to the classical monotonicity of the operators.

Definition 2.6. *An operator $\mathcal{T} : \mathcal{H} \to \mathcal{H}$ is said to be:*

(1) Strongly monotone, if there exist a constant $\alpha > 0$, such that

$$\langle \mathcal{T}\mu - \mathcal{T}\nu, \mu - \nu \rangle \geq \alpha \|\mu - \nu\|^2, \quad \forall \mu, \nu \in \mathcal{H}.$$

(2) Lipschitz continuous, if there exist a constant $\beta > 0$, such that

$$\|\mathcal{T}\mu - \mathcal{T}\nu\| \leq \beta \|\mu - \nu\|, \quad \forall \mu, \nu \in \mathcal{H}.$$

(3) Monotone, if

$$\langle \mathcal{T}\mu - \mathcal{T}\nu, \mu - \nu \rangle \geq 0, \quad \forall \mu, \nu \in \mathcal{H}.$$

(4) Pseudo monotone, if

$$\langle \mathcal{T}\mu, \nu - \mu \rangle \geq 0 \quad \Rightarrow \quad \langle \mathcal{T}\nu, \nu - \mu \rangle \geq 0, \quad \forall \mu, \nu \in \mathcal{H}.$$

Remark 2.3. *Every strongly monotone operator is a monotone operator and monotone operator is a pseudo monotone operator, but the converse is not true.*

3. Projection Method

In this section, we use the fixed point formulation to suggest and analyze some new implicit methods for solving the harmonic-like quasi variational inequalities.

Using Lemma 2.1.1, one can show that the harmonic-like quasi variational inequalities are equivalent to the fixed point problems.

Lemma 3.0.1. *The function $\mu \in \Omega(\mu)$ is a solution of the harmonic-like quasi variational inequality (7), if and only if, $\mu \in \Omega(\mu)$ satisfies the relation*

$$g(\mu) = \Pi_{\Omega(\mu)}[g(\mu) - \rho \mathcal{T}(\frac{2w\mu}{w+\mu})], \tag{23}$$

where $\Pi_{\Omega(\mu)}$ is the implicit projection operator and $\rho > 0$ is a constant.

Proof. Let $u \in \Omega(\mu)$ be a solution of the problem (7). Then

$$\langle \rho\mathcal{T}(\frac{2w\mu}{w+\mu} + g(\mu) - g(\mu), h(\nu) - g(\mu)\rangle \geq 0, \quad \forall w, \nu \in \Omega(\mu).$$

Using Lemma 2.1.1, we have

$$g(\mu) = \Pi_{\Omega(\mu)}[g(\mu) - \rho\mathcal{T}(\frac{2w\mu}{w+\mu})],$$

the required result. □

Lemma 3.0.1 implies that the harmonic-like quasi variational inequality (7) is equivalent to the fixed point problem (23). This equivalent fixed point formulation (23) will play an important role in deriving the main results.

From the equation (23), we have

$$\mu = \mu - g(\mu) + \Pi_{\Omega(\mu)}[g(\mu) - \rho\mathcal{T}(\frac{2w\mu}{w+\mu})].$$

We define the function F associated with (23) as

$$F(\mu) = \mu - g(\mu) + \Pi_{\Omega(\mu)}[g(\mu) - \rho\mathcal{T}(\frac{2w\mu}{w+\mu})], \tag{24}$$

To prove the unique existence of the solution of the problem (12), it is enough to show that the map F defined by (24) has a fixed point.

Theorem 3.1. *Let the operator g be strongly monotone with constant $\sigma > 0$ and Lipschitz continuous with constant $\zeta > 0$, respectively. If the Assumption 2.1 holds and there exists a parameter $\rho > 0$, such that*

$$\rho < \frac{1-k}{\beta}, \quad k < 1, \quad \zeta < 1, \tag{25}$$

where

$$\theta = \rho\beta + k \tag{26}$$

$$k = \sqrt{1 - 2\sigma + \zeta^2} + \eta + \zeta. \tag{27}$$

then there exists a unique solution of the problem (7).

Proof. From Lemma 3.0.1, it follows that problems (23) and (12) are equivalent. Thus it is enough to show that the map $F(u)$, defined by (24) has a fixed point.

For all $\nu \neq \mu \in \Omega(\mu)$, we have

$$\begin{aligned}
&\|F(\mu) - F(\nu)\| = \|\mu - \nu - (g(\mu) - g(\nu))\| \\
&+\Pi_{\Omega(\mu)}\|[g(\mu) - \rho\mathcal{T}(\frac{2w\mu}{w+\mu})] - \Pi_{\Omega(\nu)}[g(\nu) - \rho\mathcal{T}(\frac{2w\nu}{w+\nu})]\| \\
&= \|\nu - \mu - (g(\nu) - g(\mu))\| \\
&+\|\Pi_{\Omega(\mu)}[g(\nu) - \rho\mathcal{T}(\frac{2w\nu}{w+\nu})] - \Pi_{\Omega(\nu)}[g(\nu) - \rho\mathcal{T}(\frac{2w\nu}{w+\nu})]\| \\
&+\|\Pi_{\Omega(\nu)}[g(\nu) - \rho\mathcal{T}(\frac{2w\nu}{w+\nu})] - \Pi_{\Omega(\nu)}[g(\mu) - \rho\mathcal{T}(\frac{2w\mu}{w+\mu})]\| \\
&\leq \|\mu - \nu - (g(\mu) - g(\nu))\| \\
&+\eta\|\nu - \mu\| + \|g(\nu) - g(\mu) - \rho(\mathcal{T}(\frac{2w\nu}{w+\nu}) - \mathcal{T}(\frac{2w\mu}{w+\mu}))\| \\
&\leq \|\mu - \nu - (g(\mu) - g(\nu)\| + \eta\|\nu - \mu\| + \zeta\|\nu - \mu\| \\
&\quad +\rho\beta\|\nu - \mu\|. \qquad (28)
\end{aligned}$$

Since the operator g is strongly monotone with constants $\sigma > 0$ and Lipschitz continuous with constant $\zeta > 0$, it follows that

$$\begin{aligned}
\|\mu - \nu - (g(\mu) - g(\nu))\|^2 &\leq \|\mu - \nu\|^2 - 2\langle g(\mu) - g(\nu), \mu - \nu\rangle \\
&\quad +\zeta^2\|g(\mu) - g(\nu)\|^2 \\
&\leq (1 - 2\sigma + \zeta^2)\|\mu - \nu\|^2. \qquad (29)
\end{aligned}$$

From (28) and (29), we have

$$\begin{aligned}
\|F(\mu) - F(\nu)\| &\leq 2\left\{\sqrt{(1 - 2\sigma + \zeta^2)} + \eta + \zeta + \rho\beta\right\}\|\mu - \nu\| \\
&= \theta\|\mu - \nu\|,
\end{aligned}$$

where

$$\begin{aligned}
\theta &= \rho\beta + k \\
k &= 2\sqrt{1 - 2\sigma + \zeta^2} + \eta + \zeta.
\end{aligned}$$

From (25), it follows that $\theta < 1$, which implies that the map $F(u)$ defined by (24) has a fixed point, which is the unique solution of (7). □

The fixed point formulation (23) is applied to propose and suggest the iterative methods for solving the problem (7).

Algorithm 3.1. *For a given μ_0, compute the approximate solution $\{\mu_{n+1}\}$ by the* *iterative schemes*

$$
\begin{aligned}
y_n &= (1-\gamma_n)\mu_n \\
&\quad +\gamma_n\{\mu_n - g(\mu_n) + \Pi_{\Omega(\mu_n)}[g(\mu_n) - \rho\mathcal{T}(\frac{2w\mu_n}{w+\mu_n})]\} \quad (30)\\
z_n &= (1-\beta_n)\mu_n \\
&\quad +\beta_n\{y_n - g(y_n) + \Pi_{\Omega(y_n)}[g(y_n) - \rho\mathcal{T}(\frac{2wy_n}{w+y_n})]\} \quad (31)\\
\mu_{n+1} &= (1-\alpha_n)\mu_n \\
&\quad +\alpha_n\{z_n - g(z_n) + \Pi_{\Omega(w_n)}[z_n - \rho\mathcal{T}(\frac{2wz_n}{w+z_n})]\}. \quad (32)
\end{aligned}
$$

which are known as modified Noor iterations.

We now study the convergence analysis of Algorithm 3.1, which is the main motivation of our next result.

Theorem 3.2. *Let the operator g satisfy all the assumptions of Theorem 3.1. If the condition (25) holds, then the approximate solution $\{u_n\}$ obtained from Algorithm 3.1 converges to the exact solution $\mu \in \Omega(\mu)$ of the harmonic-like quasi variational inequality (7) strongly in $\mathcal{H}$.*

Proof. From Theorem 3.1, we see that there exists a unique solution $\mu \in \Omega(\mu)$ of the harmonic-like quasi variational inequalities (7). Let $\mu \in \Omega(\mu)$ be the unique solution of (7). Then, using Lemma 3.0.1, we have

$$
\mu = (1-\alpha_n)\mu + \alpha_n\{\mu - g(\mu) + \Pi_{\Omega(\mu)}[g(\mu) - \rho\mathcal{T}(\frac{2w\mu}{w+\mu})]\} \quad (33)
$$

$$
= (1-\beta_n)\mu + \beta_n\{\mu - g(\mu) + \Pi_{\Omega(\mu)}[g(\mu) - \rho\mathcal{T}(\frac{2w\mu}{w+\mu})]\} \quad (34)
$$

$$
= (1-\gamma_n)\mu + \gamma_n\{\mu - g(\mu) + \Pi_{\Omega(\mu)}[g(\mu) - \rho\mathcal{T}(\frac{2w\mu}{w+\mu})]\}. \quad (35)
$$

From (32),(33) and Assumption (2.1), we have

$$
\begin{aligned}
&\|\mu_{n+1} - \mu\| = \|(1-\alpha_n)(\mu_n - \mu) + \alpha_n(z_n - \mu - (g(z_n) - g(\mu))) \\
&\quad +\alpha_n\Pi_{\Omega((z_n)}[g(z_n) - \rho\mathcal{T}(\frac{2wz_n}{w+z_n})w_n] - \Pi_{(\mu)}[g(\mu) - \rho\mathcal{T}(\frac{2w\mu}{w+\mu})\}\| \\
\leq\ & (1-\alpha_n)\|\mu_n - \mu\| + \alpha_n\|z_n - \mu - (g(z_n) - g(\mu))\| \\
&\quad +\alpha_n\Pi_{\Omega((z_n)}[g(z_n) - \rho\mathcal{T}(\frac{2wz_n}{w+z_n})] - \Pi_{\Omega(w_n)}[g(\mu_n) - \rho\mathcal{T}(\frac{2w\mu}{w+\mu})\}\| \\
&\quad +\alpha_n\{\Pi_{(w_n)}[g(\mu_n) - \rho\mathcal{T}\mu] - \Pi_{\Omega(\mu)}[g(\mu) - \rho\mathcal{T}(\frac{2w\mu}{w+\mu})\}\| \\
\leq\ & (1-\alpha_n)\|\mu_n - \mu\| + \alpha_n\|z_n - \mu - (g(z_n) - g(\mu))\| \\
+\ & \alpha_n\|g(w_n) - g(\mu) - \rho(\mathcal{T}(\frac{2wz_n}{w+z_n}) - \mathcal{T}(\frac{2w\mu}{w+\mu}))|| + \alpha_n\eta\|w_n - \mu\| \\
\leq\ & (1-\alpha_n)\|\mu_n - \mu\| + \alpha_n(k+\rho\beta)||w_n - \mu\| \\
=\ & (1-\alpha_n)||u_n - \mu\| + \alpha_n\theta||w_n - \mu\|, \quad (36)
\end{aligned}
$$

where θ is defined by (26).
In a similar way, from (30) and (34), we have

$$\begin{aligned} \|z_n-\mu\| &\leq (1-\beta_n)\|\mu_n-\mu\|+2\beta_n\theta\|y_n-\mu-(g(y_n)-g(\mu))\| \\ &\quad +\beta_n\|g(y_n)-g(\mu)-\rho(\mathcal{T}(\frac{2wy_n}{w+y_n})-\mathcal{T}(\frac{2w\mu_n}{w+\mu_n})\|+\beta_n\eta\|y_n-\mu\| \\ &\leq (1-\beta_n)\|\mu_n-\mu\|+\beta_n(k+\rho\beta)\|y_n-\mu\|, \\ &\leq (1-\beta_n)\|\mu_n-\mu\|+\beta_n\theta\|y_n-\mu\|, \end{aligned} \tag{37}$$

where θ is defined by (25).
From (30) and (35), we obtain

$$\begin{aligned} \|y_n-\mu\| &\leq (1-\gamma_n)\|\mu_n-\mu\|+\gamma_n\theta\|\mu_n-\mu\| \\ &\leq (1-(1-\theta)\gamma_n)\|\mu_n-\mu\| \\ &\leq ||\mu_n-\mu||. \end{aligned} \tag{38}$$

From (37) and (38), we obtain

$$\begin{aligned} \|z_n-\mu\| &\leq (1-\beta_n)\|\mu_n-\mu\|+\beta_n\theta\|\mu_n-\mu\| \\ &= (1-(1-\theta)\beta_n)\|\mu_n-\mu\| \\ &\leq ||\mu_n-\mu||. \end{aligned} \tag{39}$$

Form the above equations, we have

$$\begin{aligned} \|\mu_{n+1}-\mu\| &\leq (1-\alpha_n)\|\mu_n-\mu\|+\alpha_n\theta\|\mu_n-\mu\| \\ &= [1-(1-\theta)\alpha_n]\|\mu_n-\mu\| \\ &\leq \prod_{i=0}^{n}[1-(1-\theta)\alpha_i]\|\mu_0-\mu\|. \end{aligned}$$

Since $\sum_{n=0}^{\infty}\alpha_n$ diverges and $1-\theta>0$, we have $\prod_{i=0}^{n}[1-(1-\theta)\alpha_i]=0$. Consequently the sequence $\{u_n\}$ convergence strongly to μ. From (38), and (39), it follows that the sequences $\{y_n\}$ and $\{w_n\}$ also converge to μ strongly in $\mathcal{H}$. This completes the proof. □

We suggest new perturbed iterative schemes for solving the harmonic-like quasi variational inequality (7).

Algorithm 3.2. *For a given μ_0, compute the approximate solution $\{\mu_n\}$ by the iter-*

ative schemes

$$\begin{aligned}
y_n &= (1-\gamma_n)\mu_n \\
&\quad +\gamma_n\{\mu_n - g(\mu_n) + \Pi_{\Omega(\mu_n)}[g(\mu_n) - \rho\mathcal{T}(\frac{2w\mu_n}{w+\mu_n})]\} + \gamma_n h_n \\
z_n &= (1-\beta_n)\mu_n \\
&\quad +\beta_n\{y_n - g(y_n) + \Pi_{\Omega(y_n)}[g(y_n) - \rho\mathcal{T}(\frac{2wy_n}{w+y_n})]\} + \beta_n f_n \\
\mu_{n+1} &= (1-\alpha_n)\mu_n \\
&\quad +\alpha_n\{z_n - g(z_n) + \Pi_{\Omega(w_n)}[g(z_n) - \rho\mathcal{T}(\frac{2wz_n}{w+z_n})\} + \alpha_n e_n,
\end{aligned}$$

where $\{e_n\}$, $\{f_n\}$, and $\{h_n\}$ are the sequences of the elements of $\mathcal{H}$ introduced to take into account possible inexact computations and $\Pi_{\Omega(\mu_n)}$ is the corresponding perturbed projection operator and the sequences $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ satisfy

$$0 \le \alpha_n, \beta_n, \gamma_n \le 1; \quad \forall n \ge 0, \quad \sum_{n=0}^{\infty} \alpha_n = \infty.$$

For $\gamma_n = 0$, we obtain the perturbed Ishikawa iterative method and for $\gamma_n = 0$ and $\beta_n = 0$, we obtain the perturbed Mann iterative schemes for solving harmonic-like quasi variational inequality (7).

For a parameter ξ, one can rewrite the (23) as

$$\mu = \mu - g(\mu) + \Pi_{\Omega(\mu)}[g((1-\xi)\mu + \xi\mu) - \rho\mathcal{T}(\frac{2w((1-\xi)\mu + \xi\mu)}{w + (1-\xi)\mu + \xi\mu})].$$

This equivalent fixed point formulation enables to suggest the following inertial method for solving the problem (7).

Algorithm 3.3. *For a given $\mu_0, \mu_1 \in \Omega(\mu)$, compute μ_{n+1} by the iterative scheme*

$$\mu_{n+1} = \mu_n - g(\mu_n) + \Pi_{\Omega(\mu_n)}[g((1-\xi)\mu_n + \xi\mu_{n-1}) - \rho\mathcal{T}\frac{2w((1-\xi)\mu_n + \xi\mu_{n-1})}{w + (1-\xi)\mu_n + \xi\mu_{n-1})}],$$

which is equivalent to the following two-step method.

Algorithm 3.4. *For given μ_0, μ_1, compute μ_{n+1} by the iterative scheme*

$$\begin{aligned}
z_n &= (1-\xi)u_n + \xi u_{n-1} \\
\mu_{n+1} &= \mu_n - g(\mu_n) + \Pi_{\Omega(\mu_n)}[g(z_n) - \rho\mathcal{T}(\frac{2wz_n}{w+z_n})].
\end{aligned}$$

Algorithm 3.4 is known as the inertial projection method.

We now suggest multi-step inertial methods for solving the harmonic-like quasi variational inequalities (7).

Algorithm 3.5. *For given μ_0, μ_1, compute μ_{n+1} by the recurrence relation*

$$
\begin{aligned}
z_n &= \mu_n - \theta_n (\mu_n - \mu_{n-1}) \\
y_n &= (1-\beta_n) z_n \\
&\quad + \beta_n \Big\{ z_n - g(z_n) + \Pi_{(\omega_n)} \Big[g(\frac{z_n+\mu_n}{2}) - \rho\mathcal{T}(\frac{2w(z_n+\mu_n)}{2w+z_n+\mu_n}) \Big] \Big\}, \\
\mu_{n+1} &= (1-\alpha_n) y_n \\
&\quad + \alpha_n \Big\{ y_n - g(y_n) + \Pi_{\Omega(y_n)} \Big[g(\frac{z_n+y_n}{2}) - \rho\mathcal{T}(\frac{2w(y_n+z_n)}{2w+y_n+z_n}) \Big] \Big\},
\end{aligned}
$$

where $\theta_n \in [0,1], \forall n \geq 1$.

Algorithm 3.5 is a three-step modified inertial method for solving harmonic-like quasi variational inequalities(7).
Similarly a four-step inertial method for solving the inverse quasi variational inequalities (7) is suggested.

Algorithm 3.6. *For given μ_0, μ_1, compute μ_{n+1} by the recurrence relation*

$$
\begin{aligned}
& z_n = \mu_n - \theta_n (\mu_n - \mu_{n-1}), \\
& y_n = (1-\gamma_n) z_n \\
& + \gamma_n \Big\{ z_n - g(\omega_n) + \Pi_{(\omega_n)} \Big[g(\frac{z_n+\mu_n}{2}) - \rho\mathcal{T}(\frac{2w(z_n+\mu_n)}{2w+z_n+\mu_n}) \Big] \Big\}, \\
& t_n = (1-\beta_n) y_n \\
& + \beta_n \Big\{ y_n - g(y_n) + \Pi_{\Omega(y_n)} \Big[g(\frac{y_n+\omega_n+\mu_n}{3}) - \rho\mathcal{T}(\frac{2w(y_n+z_n+\mu_n)}{3w+y_n+z_n+\mu_n}) \Big] \Big\}, \\
& \mu_{n+1} = (1-\alpha_n) z_n + \alpha_n \Big\{ z_n - g(z_n) \\
& + \Pi_{\Omega(z_n)} \Big[g(\frac{z_n+y_n+z_n+\mu_n}{4}) - \rho\mathcal{T}(\frac{2w(y_n+z_n+t_n+\mu_n)}{4w+y_n+z_n+t_n+\mu_n}) \Big] \Big\},
\end{aligned}
$$

where $\alpha_n, \beta_n, \gamma_n, \theta_n \in [0,1], \quad \forall n \geq 1$.

Using the technique of Noor et al. [57, 59] and Jabeen et al [26–28], one can investigate the convergence analysis of these inertial projection methods. We would like to mention these Algorithms can be viewed as the new generalizations of Noor (three-step) iterations [48, 49] for solving the general harmonic-like quasi variational inequalities. Similar multi-step hybrid iterative methods can be proposed and analyzed for solving system of harmonic-like quasi variational inequalities, which is an interesting problem.

4. Auxiliary principle technique

There are several techniques such as projection, resolvent, descent methods for solving the variational inequalities and their variant forms. None of these techniques can be applied for suggesting the iterative methods for solving the several nonlinear variational inequalities and equilibrium problems. To overcome these drawbacks, one usually applies the auxiliary principle technique, which is mainly due to Glowinski et al [21] as developed in [43, 53, 55, 64, 66, 70], to suggest and analyze some proximal point methods for solving harmonic-like quasi variational inequalities (7). The main idea involving this technique is to first consider an auxiliary problem and then to show that the solution of the auxiliary problem is the solution of the original problem by using the fixed-point approach.

For the readers convenience, we recall some basic properties of the modified Bregman distance functions, which were introduced by Noor [43] in 1992. For a arbitrary continuous operator M, we define the distance function as

$$\mathcal{M}(\nu,\mu) \quad = \quad M(\nu) - M(\mu), \nu - \mu\rangle, \forall \nu, \mu \in \Omega. \tag{40}$$

If the operator M is strongly monotone operator with constant $\zeta \geq 0$, then

$$\mathcal{M}(\nu,\mu) = M(\nu) - M(\mu), \nu - \mu\rangle \quad \geq \quad \zeta\|\nu - \mu\|^2, \quad \mu, \nu \in \Omega(\mu), \quad \forall \nu, \mu \in \Omega.$$

where ζ is the strongly monononicity constant. It is important to emphasize that various types of function M gives different modified distance function.
We apply the auxiliary principle technique involving an arbitrary operator for finding the approximate solution of the harmonic-like quasi variational inequalities (7).

For a given $\mu \in \Omega(\mu)$ satisfying (12), find $z \in \Omega(\mu)$ such that

$$\langle \rho \mathcal{T}(\frac{2w(z + \eta(\mu - z))}{w + z + \eta(\mu - z)}), h(\nu) - g(z)\rangle + \langle M(z) - M(\mu), \nu - z\rangle \geq 0, \quad \forall w, \nu \in \Omega(\mu), \tag{41}$$

where $\rho > 0, \eta \in [0,1]$ are constants and M is an arbitrary operator. The inequality (41) is called the auxiliary harmonic-like quasi variational inequality.
If $z = \mu$, then z is a solution of (7). This simple observation enables us to suggest the following iterative method for solving (7).

Algorithm 4.1. *For a given $\mu_0 \in \Omega(\mu)$, compute the approximate solution μ_{n+1} by the iterative scheme*

$$\begin{aligned} &\langle \rho \mathcal{T}(\frac{2w(\mu_{n+1} + \eta(\mu_n - \mu_{n+1}))}{w + \mu_{n+1} + \eta(\mu_n - \mu_{n+1})}), h(\nu) - g(\mu_{n+1})\rangle \\ &+\langle M(\mu_{n+1}) - M(\mu_n), \nu - \mu_{n+1}\rangle \geq 0, \quad \forall w, \nu \in \Omega(\mu) \end{aligned} \tag{42}$$

Algorithm 4.1 is called the hybrid proximal point algorithm.
Special Cases: We now discuss some special cases.
(I). For $\eta = 0$, Algorithm 4.1 reduces to

Algorithm 4.2. *For a given μ_0, compute the approximate solution μ_{n+1} by the iterative scheme*

$$\begin{aligned} &\langle \rho \mathcal{T}(\frac{2w(\mu_{n+1})}{w + \mu_{n+1}}), h(\nu) - g(\mu_{n+1})\rangle \\ &+\langle M(\mu_{n+1}) - M(\mu_n), \nu - \mu_{n+1}\rangle \geq 0, \quad \forall w, \nu \in \Omega(\mu) \end{aligned} \tag{43}$$

is called the implicit iterative methods for solving the problem (7).
(II). If $\eta = 1$, then Algorithm 4.1 collapses to

Algorithm 4.3. *For a given μ_0, compute the approximate solution μ_{n+1} by the iterative scheme*

$$\langle \rho\mathcal{T}(\frac{2w(\mu_n)}{w+\mu_n}), h(\nu) - g(\mu_{n+1})\rangle + \langle M(\mu_{n+1}) - M(\mu_n), \nu - \mu_{n+1}\rangle \geq 0, \quad \forall w, \nu \in \Omega(\mu)$$

is called the explicit iterative method.
(III). For $\eta = \frac{1}{2}$, Algorithm 4.1 becomes:

Algorithm 4.4. *For a given μ_0, compute the approximate solution μ_{n+1} by the iterative scheme*

$$\begin{aligned}&\langle \rho\mathcal{T}(\frac{2w(\mu_{n+1}+\mu_n)}{2w+\mu_{n+1}+\mu_n}), h(\nu) - g(\mu_{n+1})\\ &\quad +\langle M(\mu_{n+1}) - M(\mu_n), \nu - \mu_{n+1}\rangle \geq 0, \quad \forall w, \nu \in \Omega(\mu)\end{aligned}$$

is known as the mid-point proximal method for solving the problem (7).
For the convergence analysis of Algorithm 4.2, we need the following concepts.

Definition 4.1. *An operator $\mathcal{T}$ is said to be pseudomontone with respect to the operators g, h, if*

$$\langle \mathcal{T}(\frac{2w\mu}{w+\mu}), h(\nu) - g(\mu)\rangle \geq 0, \quad \forall w, \nu \in \Omega(\mu),$$

implies that

$$-\langle \mathcal{T}(\frac{2w\nu}{w+\nu}), h(\mu) - g(\nu)\rangle \geq 0, \quad \forall w, \nu \in \Omega(\mu).$$

Theorem 4.1. *Let the operator $\mathcal{T}$ be a pseudo-monotone with respect to the operators g, h. Let the approximate solution μ_{n+1} obtained in Algorithm 4.2 converges to the exact solution $\mu \in \Omega(\mu)$ of the problem (7). If the operator M is strongly monotone with constant $\xi \geq 0$ and Lipschitz continuous with constant $\zeta \geq 0$, then*

$$\xi\|\mu_{n+1} - \mu_n\| \leq \zeta\|\mu - \mu_n\|. \tag{44}$$

Proof. Let $\mu \in \Omega(\mu)$ be a solution of the problem (7). Then

$$-\langle \rho\mathcal{T}(\frac{2w\nu}{w+\nu}), h(\mu) - g(\nu)\rangle \geq 0, \quad \forall w, \nu \in \Omega(\mu), \tag{45}$$

since the operator $\mathcal{T}$ is a pseudo-monotone with respect to the operator g..
Takin $\nu = \mu_{n+1}$ in (45), we obtain

$$-\langle \rho\mathcal{T}(\frac{2w\mu_{n+1}}{w+\mu_{n+1}}), h(\mu) - g(\mu_{n+1})\rangle \geq 0. \tag{46}$$

Setting $\nu = \mu$ in (43), we have

$$\langle \rho\mathcal{T}(\frac{2w\mu_{n+1}}{w+\mu_{n+1}}), h(\mu) - g(\mu_{n+1})\rangle + \langle M(\mu_{n+1}) - M(\mu_n), \mu - \mu_{n+1}\rangle \geq 0. \quad (47)$$

Combining (47) and (46), we have

$$\langle M(\mu_{n+1}) - M(\mu_n), \mu - \mu_{n+1}\rangle \quad \geq \quad -\langle \rho\mathcal{T}(\frac{2w\mu_{n+1}}{w+\mu_{n+1}}), h(\mu) - g(u_{n+1})\rangle \geq 0. \quad (48)$$

From the equation (48), we have

$$\begin{aligned} 0 \quad &\leq \quad \langle M(\mu_{n+1}) - M(\mu_n), \mu - \mu_{n+1}\rangle \\ &= \quad \langle M(\mu_{n+1}) - M(\mu_n), \mu - \mu_n + \mu_n - u_{n+1}\rangle \\ &= \quad \langle M(\mu_{n+1}) - M(\mu_n), \mu - \mu_n\rangle + \langle M(\mu_{n+1} - M(\mu_n), \mu_n - \mu_{n+1}\rangle, \end{aligned}$$

which implies that

$$\langle M(\mu_{n+1} - M(\mu_n), \mu_{n+1} - \mu_n\rangle \leq \langle M(\mu_{n+1}) - M(\mu_n), \mu - \mu_n\rangle.$$

Now using the strongly monotonicity with constant $\xi > 0$ and Lipschitz continuity with constant ζ of the operator M, we obtain

$$\xi\|\mu_{n+1} - \mu_n\|^2 \leq \zeta\|\mu_{n+1} - \mu_n\|\|\mu_n - \mu\|.$$

Thus

$$\xi\|\mu_n - \mu_{n+1}\| \leq \zeta\|\mu_n - \mu\|,$$

the required result (44). □

Theorem 4.2. *Let H be a finite dimensional space and all the assumptions of Theorem 4.1 hold. Then the sequence $\{\mu_n\}_0^\infty$ given by Algorithm 4.2 converges to the exact solution $\mu \in \Omega(\mu)$ of (7).*

Proof. Let $\mu \in \Omega(\mu)$ be a solution of (12). From (44), it follows that the sequence $\{\|\mu - \mu_n\|\}$ is nonincreasing and consequently $\{u_n\}$ is bounded. Furthermore, we have

$$\xi\sum_{n=0}^{\infty}\|\mu_{n+1} - \mu_n\| \leq \zeta\|\mu_n - \mu\|,$$

which implies that

$$\lim_{n\to\infty}\|\mu_{n+1} - \mu_n\| = 0. \quad (49)$$

Let $\hat{\mu}$ be the limit point of $\{\mu_n\}_0^\infty$; whose subsequence $\{\mu_{n_j}\}_1^\infty$ of $\{\mu_n\}_0^\infty$ converges to $\hat{\mu} \in \Omega(\mu)$. Replacing μ_{n+1} by μ_{n_j} in (43), taking the limit $n_j \longrightarrow \infty$ and using (49), we have

$$\langle \rho\mathcal{T}(\frac{2w\hat{\mu}}{w+\hat{u}}), h(\nu) - g(\hat{\mu})\rangle \geq 0, \qquad \forall \nu \in \Omega(\mu),$$

which implies that $\hat{u}$ solves the problem (7) and

$$\|\mu_{n+1} - \mu\| \leq \|\mu_n - \mu\|.$$

Thus, it follows from the above inequality that $\{\mu_n\}_1^\infty$ has exactly one limit point $\hat{u}$ and

$$\lim_{n\to\infty} (\mu_n) = \hat{\mu}.$$

the required result. □

In recent years inertial type iterative methods have been applied to find the approximate solutions of variational inequalities and related optimizations. We again apply the modified auxiliary principle approach involving an arbitrary nonlinear operator to suggest some hybrid inertial proximal point schemes for solving the harmonic-like quasi variational inequalities.
For a given $\mu \in \Omega(\mu)$ satisfying (7), find $w \in \Omega(\mu)$ such that

$$\begin{aligned} &\langle \rho\mathcal{T}(\frac{2w(z+\eta(\mu-z))}{w+z+\eta(\mu-z)}), h(\nu) - g(z)\rangle \\ &\quad + \langle M(z) - M(\mu) + \alpha(\mu-\mu), , \nu - z\rangle \geq 0, \quad \forall w, \nu \in \Omega(\mu), \qquad (50) \end{aligned}$$

where $\rho > 0, \eta, \alpha \in [0,1]$ are constants and M is an arbitrary operator.
Clearly $z = \mu$, implies that w is a solution of (7). This simple observation enables us to suggest the following iterative method for solving (7).

Algorithm 4.5. *For given μ_0, μ_1, compute the approximate solution μ_{n+1} by the iterative scheme*

$$\begin{aligned} &\langle \rho\mathcal{T}(\frac{2w(\mu_{n+1}+\eta(\mu_n-\mu_{n+1})}{w+\mu_{n+1}+\eta(\mu_n-\mu_{n+1}})), h(\nu) - g(\mu_{n+1})\rangle \\ &\quad + \langle M(\mu_{n+1}) - M(\mu_n) + \alpha(\mu_n - \mu_{n-1}), \nu - \mu_{n+1}\rangle \geq 0, \quad \forall w, \nu \in \Omega(\mu). \end{aligned}$$

Algorithm 4.5 is called the hybrid proximal point algorithm for solving the harmonic-like quasi variational inequalities (7). For $\alpha = 0$, Algorithm 4.5 is exactly Algorithm 4.1.
For $M = 0$, Algorithm 4.5 reduces to the following method:

Algorithm 4.6. *For given μ_0, μ_1, compute the approximate solution μ_{n+1} by the iterative scheme*

$$\begin{aligned} &\langle \rho\mathcal{T}(\frac{2w(\mu_{n+1}+\eta(\mu_n-\mu_{n+1})}{w+\mu_{n+1}+\eta(\mu_n-\mu_{n+1}})), h(\nu) - g(\mu_{n+1})\rangle \\ &\quad + \alpha(\mu_n - \mu_{n-1}), \nu - \mu_{n+1}\rangle \geq 0, \quad \forall w, \nu \in \Omega(\mu). \end{aligned}$$

Remark 4.1. *For different and suitable choice of the parameters ρ, η, α, operators $\mathcal{T}, g, h, M$ and convex-valued sets, one can recover new and known iterative methods for solving harmonic-like quasi variational inequalities, inverse complementarity problems and related optimization problems. Using the technique and ideas of Theorem 4.1 and Theorem 4.2, one can analyze the convergence of Algorithm 4.5 and its special cases.*

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5. Dynamical Systems Technique

In this section, we consider the dynamical systems technique for solving the harmonic-likequasi variational inequalities. The projected dynamical systems associated with variational inequalities were considered by Dupuis and Nagurney [21]. It is worth mentioning that the dynamical system is a first order initial value problem. Consequently, variational inequalities and nonlinear problems arising in various branches in pure and applied sciences can now be studied via the differential equations. It has been shown [2, 19, 20, 26, 32, 47, 53–55, 59, 63–65, 74–77] that the dynamical systems are useful in developing some efficient numerical techniques for solving variational inequalities and related optimization problems. We consider some new iterative methods for solving the harmonic-like quasi variational inequalities. We investigate the convergence analysis of these new methods involving weaker conditions.

We now define the residue vector $R(\mu)$ by the relation

$$R(\mu) = \{\Pi_{\Omega(\mu)}[g(\mu) - \rho\mathcal{T}(\frac{2w\mu}{w+\mu})] - g(\mu)\}. \tag{51}$$

Invoking Lemma 3.0.1, one can easily conclude that $\mu \in \mathcal{H}$ is a solution of the problem(12), if and only if, $\mu \in \mathcal{H}$ is a zero of the equation

$$R(\mu) = 0. \tag{52}$$

We now consider a dynamical system associated with the harmonic-like quasi variational inequalities. Using the equivalent formulation (3.0.1), we suggest a class of projection dynamical systems as

$$\frac{d\mu}{dt} = \lambda\{\Pi_{\Omega(\mu)}[g(\mu) - \rho\mathcal{T}(\frac{2w\mu}{w+\mu})] - g(\mu)\}, \quad \mu(t_0) = \alpha, \tag{53}$$

where λ is a parameter. The system of type (53) is called the projection dynamical system associated with the problem (7). Here the right hand is related to the projection and is discontinuous on the boundary. From the definition, it is clear that the solution of the dynamical system always stays in $\mathcal{H}$. This implies that the qualitative results such as the existence, uniqueness and continuous dependence of the solution of (12) can be studied.
The equilibrium point of the dynamical system (64) is defined as follows.

Definition 5.1. *An element $\mu \in \mathcal{H}$, is an equilibrium point of the dynamical system (64), if,*

$$\frac{d\mu}{dx} = 0.$$

Thus it is clear that $\mu \in \mathcal{H}$ is a solution of the harmonic-like quasi variational inequality (12), if and only if, $\mu \in \mathcal{H}$ is an equilibrium point. This implies that

$\mu \in \Omega(\mu)$ is a solution of the harmonic-like quasi variational inequality (7), if and only if, $\mu \in \Omega(\mu)$ is an equilibrium point.

Definition 5.2. *[13] The dynamical system is said to converge to the solution set S^* of (53), if , irrespective of the initial point, the trajectory of the dynamical system satisfies*

$$\lim_{t\to\infty} dist(\mu(t), S^*) = 0, \tag{54}$$

where

$$dist(\mu, S^*) = inf_{\nu\in S^*}\|\mu - \nu\|.$$

It is easy to see, if the set S^* has a unique point μ^*, then (54) implies that

$$\lim_{t\to\infty} \mu(t) = \mu^*.$$

If the dynamical system is still stable at μ^* in the Lyapunov sense, then the dynamical system is globally asymptotically stable at μ^*.

Definition 5.3. *The dynamical system is said to be globally exponentially stable with degree η at μ^*, if, irrespective of the initial point, the trajectory of the system satisfies*

$$\|\mu(t) - \mu^*\| \leq u_1\|\mu(t_0) - \mu^*\|exp(-\eta(t - t_0)), \quad \forall t \geq t_0,$$

where u_1 and η are positive constants independent of the initial point.

It is clear that the globally exponentially stability is necessarily globally asymptotically stable and the dynamical system converges arbitrarily fast.

Lemma 5.0.1. *(Gronwall Lemma)[20, 34] Let $\hat{\mu}$ and $\hat{\nu}$ be real-valued nonnegative continuous functions with domain $\{t : t \leq t_0\}$ and let $\alpha(t) = \alpha_0(|t - t_0|)$, where α_0 is a monotone increasing function. If, for $t \geq t_0$,*

$$\hat{\mu} \leq \alpha(t) + \int_{t_0}^{t} \hat{\mu}(s)\hat{\nu}(s)ds,$$

then

$$\hat{\mu}(s) \leq \alpha(t)exp\{\int_{t_0}^{t} \hat{\nu}(s)ds\}.$$

We now establish that the trajectory of the solution of the projection dynamical system (53) converges to the unique solution of the harmonic-like quasi variational inequality (7). The analysis is in the spirit of Noor [47] and Xia and Wang [76, 77].

Theorem 5.1. *Let the operators $\mathcal{T}, g : H \longrightarrow H$ be Lipschitz continuous with constants $\beta > 0, \zeta > 0$ respectively. If $\lambda(\eta + 2\zeta + \rho\beta) < 1$ and Assumption 2.1 then, for each $\mu_0 \in \Omega\mu$, there exists a unique continuous solution $\mu(t)$ of the dynamical system (53) with $\mu(t_0) = \mu_0$ over $[t_0, \infty)$.*

Proof. Let

$$G(\mu) = \{\Pi_{\Omega(\mu)}[g(\mu) - \rho\mathcal{T}(\frac{2w\mu}{w+\mu})] - g(\mu)\}, \quad \forall \mu \in H.$$

where $\lambda > 0$ is a constant and $G(\mu) = \frac{d\mu}{dt}$.
$\forall \mu, \nu \in H$, we have

$$\begin{aligned}
& \|G(\mu) - G(\nu)\| \\
\leq\ & \lambda\{\Pi_{\Omega(\mu)}[g(\mu) - \rho\mathcal{T}(\frac{2w\mu}{w+\mu})] - \Pi_{\Omega(\nu)}[g(\nu) - \rho\mathcal{T}(\frac{2w\nu}{w+\nu})]\| + \|g(\mu) - g(\nu)\|\} \\
=\ & \lambda\{\|g(\mu) - g(\nu)\| + \|\Pi_{\Omega(\mu)}[g(\mu) - \rho\mathcal{T}(\frac{2w\mu}{w+\mu})] - \Pi_{\Omega(\mu)}[g(\nu) - \rho\mathcal{T}(\frac{2w\nu}{w+\nu})]\| \\
& + \|\Pi_{\Omega(\mu)}[g(\nu) - \rho\mathcal{T}(\frac{2w\nu}{w+\nu})] - \Pi_{\Omega(\nu)}[g(\nu) - \rho\mathcal{T}(\frac{2w\nu}{w+\nu})]\|\} \\
\leq\ & \lambda\{\|g(\mu) - g(\nu)\| + \eta\|\mu - \nu\| + \|g(\mu) - g(\nu) - \rho(\mathcal{T}(\frac{2w\mu}{w+\mu}) - \mathcal{T}(\frac{2w\nu}{w+\nu}))\} \\
\leq\ & \lambda\{\|g(\mu) - g(\nu)\| + \eta\|\mu - \nu\| + \{\|g(\mu) - g(\nu)\| + \rho\|\mathcal{T}(\frac{2w\mu}{w+\mu}) - \mathcal{T}(\frac{2w\nu}{w+\nu})\|\} \\
\leq\ & \lambda\{(\eta + 2\zeta + \beta\rho)\}\|\mu - \nu\|.
\end{aligned}$$

This implies that the operator $G(\mu)$ is a Lipschitz continuous with constant $\lambda\{(\eta + 2\zeta + \rho\beta)\} < 1$ and for each $\mu \in \Omega(\mu)$, there exists a unique and continuous solution $\mu(t)$ of the dynamical system (53), defined on an interval $t_0 \leq t < T_1$ with the initial condition $\mu(t_0) = \mu_0$. Let $[t_0, T_1)$ be its maximal interval of existence. Then we have to show that $T_1 = \infty$. Consider , for any $\mu \in \Omega(\mu)$,

$$\begin{aligned}
\|G(\mu)\| &= \|\frac{d\mu}{dt}\| = \lambda\|[g(\mu) - \rho\mathcal{T}(\frac{2w\mu}{w+\mu})] - g(\mu)\| \\
&\leq \lambda\{\|\Pi_{\Omega(\mu)}[g(\mu) - \rho\mathcal{T}(\frac{2w\mu}{w+\mu})] - \Pi_{\Omega(\mu)}[0]\| + \|\Pi_{\Omega(\mu)}[0] - g(\mu)\|\} \\
&\leq \lambda\{\delta\|\{g(\mu) - \rho\mathcal{T}(\frac{2w\mu}{w+\mu})\| \\
&\quad + \|\Pi_{\Omega(\mu)}[g(\mu)] - \Pi_{\Omega(\mu)}[0]\| + \|\Pi_{\Omega(\mu)}[0] - g(u)\|\} \\
&\leq \lambda\{(\rho\beta + 2 + 2\zeta)\|u\| + \|\Pi_{\Omega(\mu)}[0]\|\}.
\end{aligned}$$

Then

$$\begin{aligned}
\|\mu(t)\| &\leq \|\mu_0\| + \int_{t_0}^{t} \|\mu(s)\| ds \\
&\leq (\|\mu_0\| + k_1(t - t_0)) + k_2 \int_{t_0}^{t} \|\mu(s)\| ds,
\end{aligned}$$

where $k_1 = \lambda\|\Pi_{\Omega(\mu)}[0]\|$ and $k_2 = \delta\lambda(\rho\beta + 2 + 2\zeta)$. Hence by the Gronwall Lemma 5.0.1, we have

$$\|\mu(t)\| \leq \{\|u_0\| + k_1(t - t_0)\}e^{k_2(t-t_0)}, \quad t \in [t_0, T_1).$$

This shows that the solution is bounded on $[t_0, T_1)$. So $T_1 = \infty$. $\square$

Theorem 5.2. *If the operator $g : \mathcal{H} \longrightarrow \mathcal{H}$ is strongly monotone with constant $\sigma > 0$ and $\zeta > 0$, then the dynamical system (53) converges globally exponentially to the unique solution of the general quasi variational inequality (7).*

Proof. Since the operator g is Lipschitz continuous, it follows from Theorem 5.1 that the dynamical system (53) has unique solution $\mu(t)$ over $[t_0, T_1)$ for any fixed $\mu_0 \in H$. Let $\mu(t)$ be a solution of the initial value problem (53). For a given $\mu^* \in H$ satisfying (7), consider the Lyapunov function

$$L(\mu) = \lambda\|\mu(t) - \mu^*\|^2, \quad u(t) \in \Omega(\mu). \tag{55}$$

From (53) and (55), we have

$$\begin{aligned}
\frac{dL}{dt} &= 2\lambda\langle\mu(t) - \mu^*, \frac{d\mu}{dt}\rangle \\
&= 2\lambda\langle\mu(t) - \mu^*, \Pi_{\Omega(\mu)}[g(\mu(t)) - \rho\mathcal{T}(\frac{2w\mu(t)}{w + \mu(t)})] - g(\mu(t))\rangle \\
&= 2\lambda\langle\mu(t) - \mu^*, \Pi_{\Omega(\mu)}[g(\mu(t)) - \rho\mathcal{T}(\frac{2w\mu(t)}{w + \mu(t)})\mu(t)] - g(\mu^*) + g(\mu^*) - g(\mu(t))\rangle \\
&= -2\lambda\langle\mu(t) - \mu^*, g(\mu(t)) - g(\mu^*)\rangle \\
&\quad +2\lambda\langle\mu(t) - \mu^*, \Pi_{\Omega(\mu)}[g(\mu(t)) - \rho\mathcal{T}(\frac{2w\mu(t)}{w + \mu(t}))\mu(t)] - g(\mu^*)\rangle \\
&\leq -2\lambda\langle\rho(\mathcal{T}(\frac{2w\mu(t}{w + \mu(t})) - \mathcal{T}(\frac{2w\mu^*(t)}{w + \mu^*(t)})\mu^*), g(\mu(t)) - g(\mu^*)\rangle \\
&\quad +2\lambda\langle\mu(t) - \mu^*(t), \Pi_{\Omega(\mu)}[g(\mu(t)) - \rho\mathcal{T}(\frac{2w\mu(t)}{w + \mu(t)})] - \Pi_{\Omega(\mu)}[g(\mu^*(t)) - \rho\mu^*(t)]\rangle, \\
&\leq -2\lambda\sigma\|\mu(t) - \mu^*\|^2 + \lambda\|g(\mu(t)) - g(\mu^*)\|^2 \\
&\quad +\lambda\|\Pi_{\Omega(\mu)}[\mu(t) - \rho\mathcal{T}(\frac{2w\mu(t)}{w + \mu(t)})] - \Pi_{\Omega(\mu)}[g(\mu^*(t)) - \rho\mathcal{T}(\frac{2w\mu^*(t)}{w + \mu^*(t)})\|^2
\end{aligned} \tag{56}$$

Using the Lipschitz continuity of the operators$\mathcal{T}, g$, we have

$$\begin{aligned}
&\|\Pi_{\Omega(\mu)}[g(\mu(t)) - \rho\mathcal{T}(\frac{2w\mu(t)}{w + \mu(t)})] - \Pi_{\Omega(\mu)}[g(\mu^*(t)) - \rho\mathcal{T}(\frac{2w\mu^*(t)}{w + \mu^*(t)})]\| \\
&\quad \leq \delta\|g(\mu(t)) - g(\mu^*(t)) - \rho(\mathcal{T}(\frac{2w\mu(t)}{w + \mu(t)}) - \mathcal{T}(\frac{2w\mu^*(t)}{w + \mu^*(t)}))\| \\
&\quad \leq \delta(\zeta + \rho\beta)\|\mu(t) - \mu^*(t)\|.
\end{aligned} \tag{57}$$

From (56) and (57), we have

$$\frac{d}{dt}\|\mu(t) - \mu^*(t)\| \leq 2\xi\lambda\|\mu(t) - \mu^*(t)\|,$$

where

$$\xi = (\delta(1 + \rho\beta) - 2\sigma).$$

Thus, for $\lambda = -\lambda_1$, where λ_1 is a positive constant, we have

$$\|\mu(t) - \mu^*\| \leq \|\mu(t_0) - \mu^*\| e^{-\xi\lambda_1(t-t_0)},$$

which shows that the trajectory of the solution of the dynamical system (53) converges globally exponentially to the unique solution of the harmonic-like quasi variational inequality (7). □

We use the dynamical system (53) to suggest some iterative for solving the harmonic-like quasi variational inequalities (7).

For simplicity, we take $\lambda = 1$. Thus the dynamical system(53) becomes

$$\frac{d\mu}{dt} + g(\mu) = \Pi_{\Omega(\mu)}\big[g(\mu) - \rho\mathcal{T}(\frac{2w\mu}{w+\mu})\big], \quad \mu(t_0) = \alpha. \tag{58}$$

The forward difference scheme is used to construct the implicit iterative method. Discretizing (58), we have

$$\frac{\mu_{n+1} - \mu_n}{h} + g(\mu_n) = \Pi_{\Omega(\mu_n)}[g(\mu_n) - \rho\mathcal{T}(\frac{2w\mu_{n+1}}{w+\mu_{n+1}})], \tag{59}$$

where h is the step size.
Now, we can suggest the following implicit iterative method for solving the harmonic-like quasi variational inequality (7).

Algorithm 5.1. *For a given μ_0, compute μ_{n+1} by the iterative scheme*

$$\mu_{n+1} = \mu_n - g(\mu_n) + \Pi_{\Omega(\mu_{n+1})}\left[g(\mu_n) - \rho\mathcal{T}(\frac{2w\mu_{n+1}}{w+\mu_{n+1}}) - \frac{\mu_{n+1} - \mu_n}{h}\right],$$

This is an implicit method and is equivalent to the following two-step method.

Algorithm 5.2. *For a given μ_0, compute μ_{n+1} by the iterative scheme*

$$\begin{aligned} y_n &= \mu_n - g(\mu_n) + \Pi_{\Omega(\mu_n)}[g(\mu_n) - \rho\mathcal{T}(\frac{2w\mu_n}{w+\mu_n})] \\ \mu_{n+1} &= \mu_n - g(\mu_n) + \Pi_{\Omega(\omega_n)}\big[g(\mu_n) - \rho\mathcal{T}(\frac{2wy_n}{w+y_n}) - \frac{y_n - \mu_n}{h}\big], \end{aligned}$$

Discretizing (58), we now suggest an other implicit iterative method for solving the harmonic-like quasi variational inequality (7).

$$\frac{\mu_{n+1} - \mu_n}{h} + g(\mu_n) = \Pi_{\Omega(\mu_{n+1})}[g(\mu_{n+1}) - \rho\mathcal{T}(\frac{2w\mu_{n+1}}{w+\mu_{n+1}})], \tag{60}$$

where h is the step size.
This formulation enables us to suggest the two-step iterative method.

Algorithm 5.3. *For a given μ_0, compute μ_{n+1} by the iterative scheme*

$$\begin{aligned} y_n &= \mu_n - g(\mu_n) + \Pi_{\omega(\mu_n)}[g(\mu_n) - \rho\mathcal{T}(\frac{2w\mu_n}{w+\mu_n})\mu_n] \\ \mu_{n+1} &= \mu_n - g(\mu_n) + \Pi_{\Omega(\omega_n)}\Big[g(\omega_n) - \rho\mathcal{T}(\frac{2wy_n}{w+y_n}) - \frac{y_n-\mu_n}{h}\Big]. \end{aligned}$$

Discretizing (58), we have

$$\frac{\mu_{n+1}-\mu_n}{h} = \mu_n - g(\mu_n) + \Pi_{\Omega(\mu_{n+1})}[g(\mu_{n+1}) - \rho\mathcal{T}(\frac{2w\mu_{n+1}}{w+\mu_{n+1}})], \tag{61}$$

where h is the step size.
This helps us to suggest the following implicit iterative method for solving the problem (7).

Algorithm 5.4. *For a given μ_0, compute μ_{n+1} by the iterative scheme*

$$\begin{aligned} y_n &= \mu_n - g(\mu_n) + \Pi_{\Omega(\mu_n)}[g(\mu_n) - \rho\mathcal{T}(\frac{2w\mu_n}{w+\mu_n})] \\ \mu_{n+1} &= \mu_n - g(\mu_n) + \Pi_{\Omega(y_n)}\Big[g(y_n) - \rho\mathcal{T}(\frac{2wy_n}{w+y_n})\Big]. \end{aligned}$$

Discretizing (58), we propose another implicit iterative method.

$$\frac{\mu_{n+1}-\mu_n}{h} + g(\mu_n) = \Pi_{\Omega(\mu_{n+1})}[g(\mu_n) - \rho\mathcal{T}(\frac{2w\mu_{n+1}}{w+\mu_{n+1}})]$$

where h is the step size.

For $h = 1$, we can suggest an implicit iterative method for solving the problem (7).

Algorithm 5.5. *For a given μ_0, compute μ_{n+1} by the iterative scheme*

$$\mu_{n+1} = \mu_n - g(\mu_n) + \Pi_{\Omega(\mu_{n+1})}[g(\mu_n) - \rho\mathcal{T}(\frac{2w\mu_{n+1}}{w+\mu_{n+1}})].$$

From (58), we have

$$\frac{d\mu}{dt} + g(\mu) = \Pi_{\Omega((1-\alpha)\mu+\alpha\mu)}[g((1-\alpha)\mu+\alpha\mu) - \rho\mathcal{T}(\frac{2w(1-\alpha)\mu+\alpha\mu)}{w+(1-\alpha)\mu+\alpha\mu})], \tag{62}$$

where $\alpha \in [0,1]$ is a constant.

Discretization (62) and taking $h = 1$, we have

$$\begin{aligned} &\mu_{n+1} = \mu_n - g(\mu_n) + \Pi_{\Omega((1-\alpha)\mu_n+\alpha\mu_{n-1})}\big[g\big((1-\alpha)\mu_n+\alpha\mu_{n-1}\big) \\ &-\rho\mathcal{T}(\frac{2w((1-\alpha)\mu_n+\alpha\mu_{n-1})}{w+(1-\alpha)\mu_n+\alpha\mu_{n-1}})\big], \end{aligned} \tag{63}$$

which is an inertial type iterative method for solving the harmonic-like quasi variational inequality (7). Using the predictor-corrector techniques, we have

Algorithm 5.6. *For a given μ_0, μ_1, compute μ_{n+1} by the iterative schemes*

$$\begin{aligned} y_n &= (1-\alpha)\mu_n + \alpha\mu_{n-1} \\ \mu_{n+1} &= \mu_n - g(\mu_n) + \Pi_{\omega_n}\big[g(y_n) - \rho\mathcal{T}(\frac{2wy_n}{w+y_n})\big], \end{aligned}$$

which is known as the inertial two-step iterative method.

We now introduce the second order dynamical system associated with the harmonic-like quasi variational inequality (7). To be more precise, we consider the problem of finding $\mu \in \mathtt{H}$ such that

$$\gamma\frac{d^2\mu}{dx^2} + \frac{d\mu}{dx} = \lambda\{\Pi_{\Omega(\mu)}[g(\mu) - \rho\mathcal{T}(\frac{2w\mu}{w+\mu})] - g(\mu)\}, \quad \mu(a) = \alpha, \quad \mu(b) = \beta, \tag{64}$$

where $\gamma > 0, \lambda > 0$ and $\rho > 0$ are constants. We would like to emphasize that the problem (64) is indeed a second order boundary vale problem. In a similar way, we can define the second order initial value problem associated with the dynamical system.

The equilibrium point of the dynamical system (64) is defined as follows.

Definition 5.4. *An element $\mu \in \mathcal{H}$, is an equilibrium point of the dynamical system (64), if,*

$$\gamma\frac{d^2\mu}{dx^2} + \frac{d\mu}{dx} = 0.$$

Thus it is clear that $\mu \in \mathcal{H}$ is a solution of the harmonic-like quasi variational inequality (7), if and only if, $\mu \in \mathcal{H}$ is an equilibrium point.
From (64), we have

$$g(\mu) = \Pi_{\Omega(\mu)}[g(\mu) - \rho\mathcal{T}(\frac{2w\mu}{w+\mu})].$$

Thus, we can rewrite (64) as follows:

$$g(\mu) = \Pi_{\Omega(\mu)}\big[g(\mu) - \rho\mathcal{T}(\frac{2w\mu}{w+\mu}) + \gamma\frac{d^2\mu}{dx^2} + \frac{d\mu}{dx}\big]. \tag{65}$$

For $\lambda = 1$, the problem (64) is equivalent to finding $\mu \in \Omega$ such that

$$\gamma\ddot{\mu} + \dot{\mu} + g(\mu) = \Pi_{\Omega(\mu)}\big[g(\mu) - \rho\mathcal{T}(\frac{2w\mu}{w+\mu})\big], \quad \mu(a) = \alpha, \quad \mu(b) = \beta. \tag{66}$$

The problem (66) is called the second dynamical system, which is in fact a second order boundary value problem. This interlink among various fields of mathematical and engineering sciences is fruitful in developing implementable numerical methods for finding the approximate solutions of the harmonic-like quasi variational inequalities. We discretize the second-order dynamical systems (66) using central finite difference and backward difference schemes to have

$$\gamma\frac{\mu_{n+1}-2\mu_n+\mu_{n-1}}{h^2}+\frac{\mu_n-\mu_{n-1}}{h}+g(\mu_n)=P_{\Omega(\mu_n)}[g(\mu_n)-\rho\mathcal{T}(\frac{2w\mu_{n+1}}{w+\mu_{n+1}})], \quad (67)$$

where h is the step size.
If $\gamma=1, h=1$, then, from equation(67) we have

Algorithm 5.7. *For a given μ_0, compute μ_{n+1} by the iterative scheme*

$$\mu_{n+1}=\mu_n+g(\mu_n)+\Pi_{\Omega(\mu_n)}[g(\mu_n)-\rho\mathcal{T}(\frac{2w\mu_{n+1}}{w+\mu_{n+1}})],$$

which is the extragradient method for solving the harmonic-like quasi variational inequalities (7). Algorithm 5.7 is an implicit method. To implement the implicit method, we use the predictor-corrector technique to suggest the method.

Algorithm 5.8. *For given μ_0, μ_1, compute μ_{n+1} by the iterative scheme*

$$\begin{aligned} y_n &= (1-\theta_n)\mu_n+\theta_n\mu_{n-1} \\ \mu_{n+1} &= \mu_n-g(\mu_n)+\Pi_{\Omega(\mu_n)}[g(\mu_n)-\rho\mathcal{T}(\frac{2wy_n}{w+y_n})], \end{aligned}$$

is called the two-step inertial iterative method, where $\theta_n\in[0,1]$ is a constant.

We discretize the second-order dynamical systems (53) using central finite difference and backward difference schemes to have

$$\gamma\frac{\mu_{n+1}-2\mu_n+\mu_{n-1}}{h^2}+\frac{\mu_n-\mu_{n-1}}{h}+g(\mu_{n+1})=\Pi_{\Omega(\mu_{n+1})}\big[g(\mu_n)-\rho\mathcal{T}(\frac{2w\mu_{n+1}}{w+\mu_{n+1}})\big],$$

where h is the step size.
Using this discrete form, we can suggest the following an iterative method for solving the harmonic-like quasi variational inequalities (7).

Algorithm 5.9. *For given μ_0, μ_1, compute μ_{n+1} by the iterative scheme*

$$\begin{aligned} &\mu_{n+1}=\mu_n-g(\mu_{n+1}) \\ &+\Pi_{\Omega(\mu_n)}\big[g(\mu_{n+1})-\rho\mathcal{T}(\frac{2w\mu_{n+1}}{w+\mu_{n+1}})-\gamma\frac{\mu_{n+1}-2\mu_n+\mu_{n-1}}{h^2}+\frac{\mu_n-\mu_{n-1}}{h}\big]. \end{aligned}$$

Algorithm 5.9 is called the hybrid inertial proximal method for solving the harmonic-like quasi variational inequalities and related optimization problems. This is a new proposed method.
Note that, for $\gamma=0$, Algorithm 5.9 reduces to the following iterative method.

Algorithm 5.10. *For given μ_0, μ_1, compute μ_{n+1} by the iterative scheme*

$$\mu_{n+1} = \mu_n - g(\mu_{n+1}) + \Pi_{\Omega(\mu_{n+1})}\big[g(\mu_{n+1}) - \rho\mathcal{T}(\frac{2w\mu_{n+1}}{w+\mu_{n+1}}) + \frac{\mu_n - \mu_{n-1}}{h}\big],$$

which is called the inertial double projection method.

We now consider the third order dynamical systems associated with the harmonic-likequasi variational inequalities of the type (7). To be more precise, we consider the problem of finding $\mu \in \mathcal{H}$, such that

$$\begin{aligned}
&\gamma\frac{d^3\mu}{dt^3} + \zeta\frac{d^2\mu}{dt^2} + \xi\frac{d\mu}{dt} + g(\mu) = \Pi_{\Omega(\mu)}[g(\mu) - \rho\mathcal{T}(\frac{2w\mu}{w+\mu})],\\
&u(a) = \alpha, \dot{\mu}(a) = \beta, \dot{\mu}(b) = 0
\end{aligned} \tag{68}$$

where $\gamma > 0, \zeta, \xi$ and $\rho > 0$ are constants. Problem 68) is called third order dynamical system associated with harmonic-like quasi variational inequalities (7).
The equilibrium point of the dynamical system (68) is defined as follows.

Definition 5.5. *An element $\mu \in \mathcal{H}$, is an equilibrium point of the dynamical system (64), if,*

$$\gamma\frac{d^3\mu}{dt^3} + \zeta\frac{d^2\mu}{dt^2} + \xi\frac{d\mu}{dt} = 0.$$

Thus it is clear that $\mu \in \mathcal{H}$ is a solution of the harmonic-like quasi variational inequality (7), if and only if, $\mu \in \mathcal{H}$ is an equilibrium point.
Consequently, the problem (53) can be written as

$$g(\mu) = \Pi_{\Omega(\mu)}\Big[g(\mu) - \rho\mathcal{T}(\frac{2w\mu}{w+\mu}) + \gamma\frac{d^3\mu}{dt^3} + \zeta\frac{d^2\mu}{dt^2} + \xi\frac{d\mu}{dt}\Big]. \tag{69}$$

We discretize the third-order dynamical systems (68) using central finite difference and backward difference schemes to have

$$\begin{aligned}
&\gamma\frac{u_{n+2} - 2u_{n+1} + 2u_{n-1} - u_{n-2}}{2h^3} + \zeta\frac{u_{n+1} - 2u_n + u_{n-1}}{h^2}\\
&+\xi\frac{3\mu_n - 4\mu_{n-1} + \mu_{n-2}}{2h} + g(\mu_n) = \Pi_{\Omega(\mu_n)}[g(\mu_n) - \rho\mathcal{T}(\frac{2w\mu_{n+1}}{w+\mu_{n+1}})],
\end{aligned} \tag{70}$$

where h is the step size.
If $\gamma = 1, h = 1, \zeta = 1, \xi = 1$, then, from equation(70) after adjustment, we have

Algorithm 5.11. *For a given μ_0, μ_1, compute u_{n+1} by the iterative scheme*

$$u_{n+1} \quad = \quad \mu_n - g(\mu_n) + \Pi_{\Omega(\mu_n)}\big[g(\mu_n) - \rho\mathcal{T}(\frac{2w\mu_{n+1}}{w+\mu_{n+1}})\big],$$

which is an implicit iterative methods and is equivalent to three step inertial iterative method for solving the harmonic-like quasi variational inequalities (7).

Algorithm 5.12. *For a given μ_0, μ_1, compute u_{n+1} by the iterative scheme*

$$\begin{aligned} y_n &= (1-\theta_n)\mu_n + \theta_n \mu_{n-1} \\ z_n &= \mu_n - g(\mu_n) + \Pi_{\Omega(\mu_n)}[g(\mu_n) - \rho\mathcal{T}(\frac{2wy_n}{w+y_n})] \\ t_n &= \mu_n - g(\mu_n) + \Pi_{\Omega(\mu_n)}[g(\mu_n) - \rho\mathcal{T}(\frac{2wy_n}{w+n_n})] \\ u_{n+1} &= \mu_n - g(\mu_n) + \Pi_{\Omega(\mu_n)}\big[g(\mu_n) - \rho\mathcal{T}(\frac{2wt_n}{w+t_n})\big], \end{aligned}$$

which can be viewed as a novel extension of the well kown Noor (three step) iterations.

Remark 5.1. *For appropriate and suitable choice of the operators $g, \mathcal{T}$ convex-valued set, parameters and the spaces, one can suggest a wide class of implicit, explicit and inertial type methods for solving harmonic-like quasi variational inequalities and related optimization problems. Using the techniques and ideas of Noor et al [53, 54, 64, 65], one can discuss the convergence analysis of the proposed methods.*

6. Generalizations and Applications

We would like to mention that some of the results obtained and presented in this paper can be extended for more general harmonic-like quasi variational inequalities. To be more precise, let $\mathcal{T}, \mathcal{A}, g, h : \mathcal{H} \longrightarrow \mathcal{H}$ be the operators. We consider the problem of finding $\mu \in \Omega(\mu)$ such that

$$\langle \rho(\mathcal{T}(\frac{2w\mu}{w+\mu}) + \rho\mathcal{A}(\frac{2w\mu}{w+\mu}), h(\nu) - g(\mu)\rangle \geq 0, \quad \forall w, \nu \in \Omega(\mu), \tag{71}$$

where ρ is a constant, is called the general harmonic-like quasi variational inequality involving the sum(difference) of two operators.
If $w = \mu, h = 1 = g, \Omega(\mu) = \Omega$, then the problem (71) reduces to finding $\mu \in \Omega$ such that

$$\langle \mathcal{T}\mu + \mathcal{A}\mu, \nu - \mu\rangle \geq 0, \quad \forall \nu \in \Omega, \tag{72}$$

is called the mlildy nonlinear variational inequalities, which was introduced and studied by Noor [37] in 1975. For appropriate and suitable choice of the operator, these mildly nonlinear variational inequalities inlclude hemivariational inequalities, difference of two operators and system of absolute vlue variational inequalities.
For $\mathcal{A}(\frac{2w\mu}{w+\mu}) = \mathcal{A}|\frac{2w\mu}{w+\mu}|$, the problem (71) reduces to finding $\mu \in \Omega(\mu)$, such that

$$\langle \rho\mathcal{T}(\frac{2w\mu}{w+\mu}) + \rho\mathcal{A}|\frac{2w\mu}{w+\mu}|, h(\nu) - g(\mu)\rangle \geq 0, \quad \forall \nu \in \Omega(\mu), \tag{73}$$

is called the absolute value harmonic-like quasi variational inequality. Note that, if $\mathcal{A} = 0, \quad h = g$, then the problem (6) is equivalent to find $\mu \in \Omega(\mu)$, such that

$$\langle \rho\mathcal{T}(\frac{2w\mu}{w+\mu}), g(\nu) - g(\mu) \geq 0 \quad \forall w, \nu \in \Omega(\mu),$$

which is the harmonic-like quasi variational inequality.
Using Lemma 3.0.1, one can prove that the problem (71) is equivalent to finding $\mu \in \Omega(\mu)$ such that

$$g(u) = \Pi_{\Omega(\mu)}[g(\mu) - \rho(\mathcal{T}(\frac{2w\mu}{w+\mu}) + \rho\mathcal{A}(\frac{2w\mu}{w+\mu})] \tag{74}$$

which can be written as

$$\mu = \mu - g(\mu) + \Pi_{\Omega(\mu)}[g(\mu) - \rho(\mathcal{T}(\frac{2w\mu}{w+\mu}) + \rho\mathcal{A}(\frac{2w\mu}{w+\mu})].$$

Thus one can consider the mapping Φ associated with the problem (71) as

$$\Phi(\mu) = \mu - g(\mu) + \Pi_{\Omega(\mu)}[g(\mu) - \rho(\mathcal{T}(\frac{2w\mu}{w+\mu}) + \rho\mathcal{A}(\frac{2w\mu}{w+\mu})],$$

which can be used to discuss the uniqueness of the solution and to propose iterative methods for the problem (71).
From (71) and (6, it follows that the general harmonic–like quasi variational inequalities are equivalent to the fixed problems. Consequently, all results obtained for the problem (7) continue to hold for the problem (71) with suitable modifications and adjustments. We would like to mention that one can obtain various classes of harmonic-like quasi variational inequalities for appropriate and suitable choices of the operators $\mathcal{T}, \mathcal{A}, g, h$, and convex-valued set $\Omega(\mu)$.
Applying the technique and idea of this paper, similar results can be established for solving system of harmonic-like quasi variational inequalities considered with appropriate modifications. The development of efficient implementable numerical methods for solving the multivalued general quasi variational inequalities and nonconvex optimization problems requires further efforts.

Conclusion

In this paper, we have used the equivalence between the harmonic-like quasi variational inequalities and fixed point problems to suggest some new multi step multi-step iterative methods for solving the quasi variational inequalities. These new methods include extragradient methods, modified double projection methods and inertial type are suggested using the techniques of projection method, modified auxiliary techniques and dynamical systems. Convergence analysis of the proposed method is discussed for monotone operators. It is an open problem to compare these proposed methods with other methods. Applying the technique and ideas of Ashish [4], Ashish et. al. [5–7], Cho et al. [13], Kwuni et al. [28], Natarajan et al. [33] and Yadav et al. [79], can one explore the Julia set and Mandelbrot set in Noor orbit using the new modified Noor (three step) iterations for solving harmonic-like quasi variational inequalities in the fixed point theory and will continue to inspire further research in fractal geometry, chaos theory, coding, number theory, spectral geometry, dynamical systems, complex analysis, nonlinear programming, graphics and computer aided design. This is an open problem, which deserves further research efforts. We have shown that the harmonic-like quasi variational inequalities are equivalent to the strongly general variational inequalities under suitable conditions of the convex-valued set. Applications of the fuzzy set theory [44], stochastic control[9], quantum calculus, fractal, logistic map [79], fractional and random traffic equilibrium [9], data analysis [14, 34] and traffic assignment [73] can be found in many branches of mathematical and engineering sciences including artificial intelligence, computer science, control engineering, management science, operations research, green energy [33] and variational inequalities. One may explore these aspects of the harmonic-like quasi variational inequalities and their variant forms.

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